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Eigenvalue Distribution in the Self-Dual Non-Hermitian Ensemble

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We consider an ensemble of self-dual matrices with arbitrary complex entries. This ensemble is closely related to a previously defined ensemble of anti-symmetric matrices with arbitrary complex entries. We study the two-level correlation functions numerically. Although no evidence of non-monotonicity is found in the real space correlation function, a definite shoulder is found. On the analytical side, we discuss the relationship between this ensemble and the $\beta = 4$ two-dimensional one-component plasma, and also argue that this ensemble, combined with other ensembles, exhausts the possible universality classes in non-hermitian random matrix theory. This argument is based on combining the method of hermitization of Feinberg and Zee with Zirnbauer's classification of ensembles in terms of symmetric spaces.

KEY WORDS: Non-hermitian; random matrix; one-component plasma.

1. INTRODUCTION AND CLASSIFICATION

There are ten known universality classes of hermitian random matrices. Dyson[(1)] proposed the existence of three symmetry classes, depending on spin and the existence of time reversal symmetry. These give the three classes known as Gaussian Unitary, Orthogonal, and Symplectic (GUE, GOE, GSE). Another three ensembles are the chiral Gaussian ensembles (chGUE, chGOE, chGSE).[(2)] These ensembles are of relevance to low

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energy QCD. Altland and Zirnbauer introduced four more ensembles which can appear in superconducting systems.[3] Finally, Zirnbauer demonstrated a relationship between the different classes of random matrix theory and symmetric spaces, and from this argued that the ten distinct known universality classes exhausted all possible universality classes,[4] subject to the qualification that some disordered systems have a transfer matrix group that is not semi-simple, and cannot be represented by a single element in this classification.[5]

In this section we discuss various universality classes of non-Hermitian random matrices, including the ensemble of arbitrary self-dual matrices, the subject of this paper. We mention the concept of weak non-Hermiticity, but do not consider it further in this paper. We argue that the various classes of non-Hermitian matrices, the self-dual ensemble and four others, exhaust all possible universality classes. Finally, possible applications of the self-dual ensemble are dealt with, including relations with the one-component plasma. In Section II, we further discuss the relationship with the one-component plasma. In Section III, numerical results for the self-dual ensemble are discussed, in particular the eigenvalue density as a function of radius and the two-eigenvalue correlation functions.

Several ensembles of non-Hermitian random matrices are common in the literature. Ginibre[6] introduced three classes of such matrices. One is an ensemble of matrices with arbitrary complex elements, one an ensemble with arbitrary real elements, and the third an ensemble with arbitrary real quaternion elements. Another ensemble of non-Hermitian matrices is an ensemble of complex, symmetric matrices.[7] This ensemble arises particularly in problems of open quantum systems. This gives a total of four known universality classes. For each of these ensembles, there exists a weakly non-Hermitian version of that ensemble. This idea of weak non-Hermiticity was introduced by Fyodorov *et al.*[9] In this case the anti-Hermitian part of the matrix is small; we only consider strongly non-Hermitian matrices in the present paper and do not consider weakly non-Hermitian matrices, even though they are the most relevant for scattering problems.

The strongly non-Hermitian ensembles can be obtained from a general three parameter family of non-Hermitian matrices introduced by Fyodorov *et al.*[8] This family includes parameters measuring the strength of the real and imaginary, symmetric and anti-symmetric parts of the matrix. By adjusting the parameters, one can obtain various ensembles. One possiblity, which does not appear to have been considered much, is an ensemble of anti-symmetric matrices with arbitrary complex elements.

Now, let us show that this ensemble is equivalent to an ensemble of self-dual matrices with arbitrary complex elements; this is the ensemble considered in this paper. Let A be an arbitrary anti-symmetric matrix. Let Z be the matrix given by

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \cdots \end{pmatrix} \tag{1}$$

Then, $Z^{\mathrm{T}} = -Z$ and $Z^2 = -1$. Let $M = ZA$. It is trival to verify that $ZM^{\mathrm{T}}Z = -M$. So, M is self-dual. The advantage of using self-dual matrices instead of anti-symmetric matrices is that self-dual matrices have pairs of equal eigenvalues while anti-symmetric matrices have pairs of opposite eigenvalues; this makes the correlation functions clearer. When choosing matrices from the ensemble, we will use Gaussian weight

$$e^{-1/2\,\mathrm{Tr}(M^{\dagger}M)} \tag{2}$$

Given these five classes, the 3 ensembles of Ginibre as well as the ensembles of symmetric non-Hermitian and self-dual non-Hermitian, let us ask whether all possible universality classes of strongly non-Hermitian random matrices have been found. Feinberg and Zee introduced the method of hermitization for non-Hermitian matrices.[10] A similar technique was used by Efetov.[11] The basic idea is to take a non-Hermitian matrix $M - E$, where E is a complex number, and form the Hermitian matrix

$$H = \begin{pmatrix} M_h - E_R & M_a - iE_I \\ M_a^{\dagger} + iE_I & -M_h + E_R \end{pmatrix} \tag{3}$$

where M_h is the Hermitian component of M and M_a is the anti-Hermitian component of M and E_R and E_I are the real and imaginary components of E. Equivalently, one can form the Hermitian matrix

$$H = \begin{pmatrix} 0 & M - E \\ M^{\dagger} - \bar{E} & 0 \end{pmatrix} \tag{4}$$

From the zero eigenvalues of H, one may extract the zero eigenvalues of $M - E$. So, to each universality class of non-Hermitian random matrices, there corresponds a universality class of Hermitian random matrices.

If we hermitize the three non-Hermitian ensembles introduced by Ginibre, we obtain the three chiral ensembles (chGUE, chGOE, chGSE).

The relation with the chiral ensembles is most clear using Eq. (4), instead of Eq. (3). If we hermitize the ensemble of symmetric, complex matrices we obtain the ensemble with symmetry class CI, according to the nomenclature of Altland and Zirnbauer. If we hermitize the ensemble of self-dual complex matrices, we obtain the ensemble with symmetry class DIII. Here the relation with the Hermitian ensembles is most clear using Eq. (3). The other five classes of hermitian random matrices cannot be obtained by hermitizing a non-Hermitian ensemble: the GOE, GUE, and GSE classes lack the needed block structure, while the C and D ensembles lack the symmetry that relates the elements in the upper left and lower right blocks. This suggests that all possible universality classes of non-Hermitian matrices have been obtained.

One interest in the ensemble of self-dual complex matrices is experimental. Consider a quantum system with Hamiltonian H. If we couple this system to the outside with some number of channels, one obtains an effective non-Hermitian Hamiltonian, reflecting the decay of particles out of the system. For systems with preserved time-reversal invariance, and no spin orbit coupling,(7) then complex, symmetric matrices are appropriate to describe the statistics of resonances in the complex energy plane. In the physical case, the imaginary part of the effective non-Hermitian Hamiltonian is always positive definite. However, for the case of an open system coupled to a large number of channels, a reasonable starting point for the eigenvalue statistics is the Gaussian ensemble of symmetric, complex matrices with a constant imaginary part added. If instead the open quantum system describes a particle with spin, then H will be self-dual, and for the open quantum system with a large number of channels, the ensemble considered in this paper will be more appropriate. Specifically, we have in mind an electron in a quantum dot, with spin orbit scattering, and strong coupling through external leads to reservoirs.

Another interest is theoretical, considering the relationship of this ensemble to the $\beta=4$ one-component plasma in two-dimensions. Although the level distribution in the ensemble differs from the distribution of charges in the plasma, there are some close relations between the two, discussed more in the next section.

It is known that the ensemble of matrices with arbitrary complex elements is equivalent to the $\beta=2$ plasma. The correlation function of the $\beta=2$ system is monotonic, with Gaussian decay. From perturbative calculations,(12) it has been suggested that, for $\beta > 2$, the two-level correlation function becomes non-monotonic, indicating the appearance of short-range order. This makes it very interesting to examine the correlation function of the ensemble of self-dual matrices, although no significant sign of any non-monotonicity is found here in the numerical calculations.

Numerical calculations on the one-component plasma[13] suggest that there is a phase transition at $\beta \approx 144$; so, any order that exists for $\beta = 4$ must be short range. An exact study for finite number of particles[14] showed non-monotonicity of the correlation functions for $\beta = 4$, 6. Even for $\beta = 4$ there is a definite peak in the correlation function.

II. $\beta = 4$ ONE-COMPONENT PLASMA

Consider a system of N particles, located at positions z_i, with partition function

$$\int dz_i \, d\bar{z}_i \prod_{i=1}^{N} e^{-|z_i|^2} \prod_{i<j} e^{\beta \log(|z_i - z_j|)} \tag{5}$$

This defines the two-dimensional one-component plasma. For $\beta = 4$, there exists some relation between this system and the ensemble considered here.

First, the density of the plasma, ρ, is equal to $\frac{1}{2\pi}$, where the density is measured in charges per unit area. The plasma has constant charge density ρ in a disc about the origin, and vanishing charge density outside. The eigenvalue density in the self-dual ensemble is the same as the charge density in the one-component plasma, as found numerically in the next section, and as can be shown with a replica or SUSY technique (the method of hermitization provides a way of performing SUSY calculations on non-Hermitian sysytems[11]).

Second, there exists a relationship between the joint probability distribution of the eigenvalues of M and the probability distribution of charges in the one-component plasma. The j.p.d. of the eigenvalues of M *is different from the charge distribution in the plasma*, but we will argue that for widely separated eigenvalues the j.p.d. of the eigenvalues behaves the same as the probability distribution of the charges. This then provides the explanation for the equivalence of the eigenvalue and charge densities.

Let M be a matrix within the ensemble of self-dual, complex matrices. We can write M as $M = X \Lambda X^{-1}$, where Λ is a diagonal matrix of eigenvalues of M. The eigenvalues of Λ exist in pairs, with $[\Lambda, Z] = 0$. The requirement that M be self-dual is equivalent to the requirement that $X^T Z = Z X^{-1}$ and $Z(X^{-1})^T = XZ$; if this constraint on X holds it is easy to verify that $Z M^{\mathrm{T}} Z = -M$.

If we were to impose the additional constraint on X that X be unitary, then we would find that X must be an element on the symplectic group. In this case, with X in the symplectic group, the matrix M must be normal, such that $[M, M^\dagger] = 0$. In this case the distribution of eigenvalues of M exactly matches the charge distribution in the $\beta = 4$ plasma.

In the general case, M is not normal and X is not unitary, and the distribution of eigenvalues of M will be different from the charge distribution of the plasma. Still, consider a situation in which we fix Λ and integrate over X, with Gaussian weight $e^{-1/2\,\mathrm{Tr}(M^\dagger M)}$. This is how one obtains the j.p.d. of the eigenvalues.

The measure $[\mathrm{d}M]$ on matrices M is equivalent to the measure $[\mathrm{d}\lambda_i][\mathrm{d}X]\prod_{i<j}|\lambda_i-\lambda_j|^8$. The j.p.d. of the eigenvalues is defined by

$$\prod_{i<j}|\lambda_i-\lambda_j|^8\int[\mathrm{d}X]\,e^{-1/2\,\mathrm{Tr}(M^\dagger M)} \tag{6}$$

with $M=X\Lambda X^{-1}$. The Gaussian weight, $e^{-1/2\,\mathrm{Tr}(M^\dagger M)}$, will depend on X. It will be greatest when X is chosen to be symplectic, so that M is normal. If the eigenvalues of Λ are well separated, then the exponential in the Gaussian weight will be large, and we can evaluate the integral by a saddle point method, as discussed in more detail in an Appendix: we will restrict our attention to a saddle point manifold of matrices M which are normal, as well as weak fluctuations .away from this saddle point manifold. If we parametrize the fluctuations away from the saddle point manifold and then treat these fluctuations in a Gaussian approximation, valid when the eigenvalue separation is large, we obtain that the j.p.d. for the self-dual ensemble is equal to, in this particular approximation,

$$\prod_{i=1}^{N}e^{-|z_i|^2}\prod_{i<j}(|z_i-z_j|)^4\prod_{i=1}^{N}dz_i\,d\bar{z}_i \tag{7}$$

up to a constant factors. This is, of course, the same as the probability distribution of the charges in the one-component plasma at $\beta=4$.

In general, we expect that for well separated eigenvalues, the level repulsion in the self-dual ensemble will match the charge repulsion in the plasma; it is only the short distance interaction that will be different. Further, it may be shown explicitly by calculations on small matrices that the short distance interactions in the j.p.d. for the self-dual ensemble cannot be written as a product of two-body terms.

We have not been able to find a simple, exact expression for the j.p.d. of the eigenvalues. By calculations on 4-by-4 matrices M, we can show that the two-eigenvalue interaction is different from that in the one-component plasma. By proceeding to 6-by-6 matrices, we can see that there are also three-eigenvalue interactions. It appears that in general there are k-eigenvalue interactions. The mathematical reason for the difficulty lies in the integral over X, above. This integral is not Gaussian, and we have no easy way of performing it. The ability to perform the analogous integral for the ensemble of complex matrices was essential to Ginibre's computation of the j.p.d. in that case.[6]

Given these similarities, one might hope that the correlation functions of the self-dual ensemble will shed some light on correlations within the plasma. In the next section, we discuss a numerical investigation of the self-dual ensemble.

III. NUMERICS

Mathematica was used to generate 4940 600-by-600 self-dual matrices. The matrices were chosen with Gaussian weight $e^{-1/2\,\mathrm{Tr}(M^\dagger M)}$ as in Eq. (2). The matrices have 300 pairs of eigenvalues. A picture of these eigenvalues for a typical matrix is shown in Fig. 1.

The eigenvalue density as a function of radius is shown in Fig. 2. The density obeys the circular law:[6, 16] it is nonvanishing and roughly constant within a disc, and vanishing outside. For the $\beta=4$ one component plasma, with a confining potential $e^{-\bar{z}z}$ (see Eq. (5), the expected density of particles per unit area, from the circular law, is $\frac{1}{2\pi}$. The single particle eigenvalue density observed numerically for the self-dual matrices agrees with this result; note that since eigenvalues come in pairs, then we expect $e^{-\bar{z}z}$ to be the confining potential that corresponds to the weight of Eq. (2) as each eigenvalue in the pair contributes a factor of $e^{\bar{z}z/2}$.

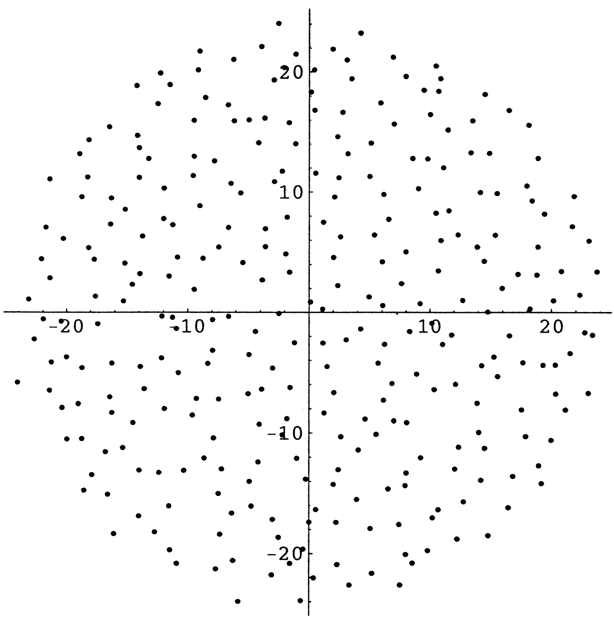

Fig. 1. Plot of eigenvalues for a typical 600-by-600 matrix.

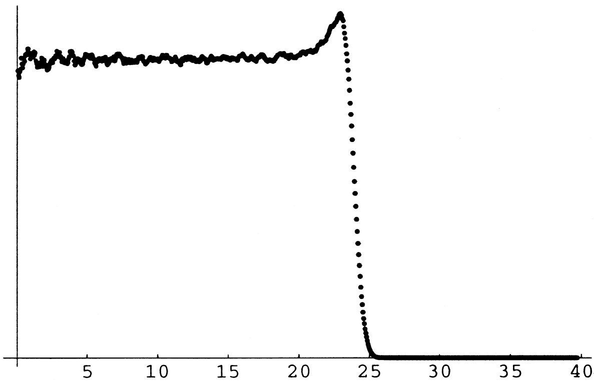

Fig. 2. Average eigenvalue density as a function of radius.

One interesting feature of Fig. 2 is that the eigenvalue density near the edge rises before dropping. This is probably a result of the short-distance behavior of the two-eigenvalue interaction. From calculations on 4-by-4 matrices, we can see that the repulsion between eigenvalues is weaker than $|z_i - z_j|^4$ for small separations. Compared to the one-component plasma case, this reduces the repulsion that the charges on the edge feel due to the charges closer to $z=0$, and enables the charges on the edge to move to smaller radii, creating a peak in the density.

Near $z=0$, there is a clear modulation in the eigenvalue density. From numerical calculations on finite-size Hermitian random matrices,[15] it is common to see modulations in the average density, as a result of level repulsion. These modulations can be seen at all energies, but disappear as the size of the matrix tends to infinity. For the non-Hermitian case it is not surprising that we also see modulations in the density; the reason they are strongest near $z=0$ is that the total number of eigenvalues at a given $|z|$ increases linearly with $|z|$, and so away from $z=0$ the modulations become smeared.

The two-level correlation function is shown in Figs. 3 and 4. In Fig. 3, we look at all eigenvalues within a distance of 6 or less from the origin, and plot the probability to find another eigenvalue at given distance from the first eigenvalue. In Fig. 4, to reduce effects due to the finite size of the matrix M, we require that the first eigenvalue lie within a distance of 3.5 or less from the origin. No significant differences are found between Fig. 3 and Fig. 4, indicating that the effects due to the finite size of M are small even in Fig. 3.

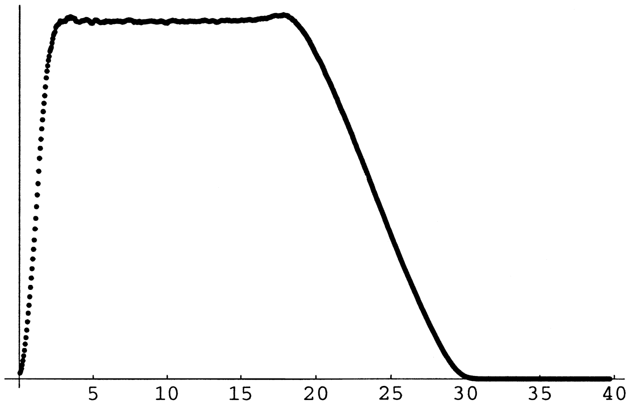

Fig. 3. Average two level correlation function. See text.

One can see finite size effects in both Figs. 3 and 4 for large distances. The correlation function rises for distances of around 20. This is simply due to the rise in eigenvalue density near the edge, as shown in Fig. 2, and has no deep meaning.

Looking at Figs. 3 and 4, there is a definite "shoulder" at a distance of slightly less than 3. There is no definite sign of any non-monotonicity; certainly, if there is any peak in the correlation function near the shoulder, it is much smaller than the peak found in the $\beta=4$ plasma.[14] As a quick estimate of the expected spacing between levels, assume that the levels formed a perfect hexagonal lattice, so that they are very ordered, and

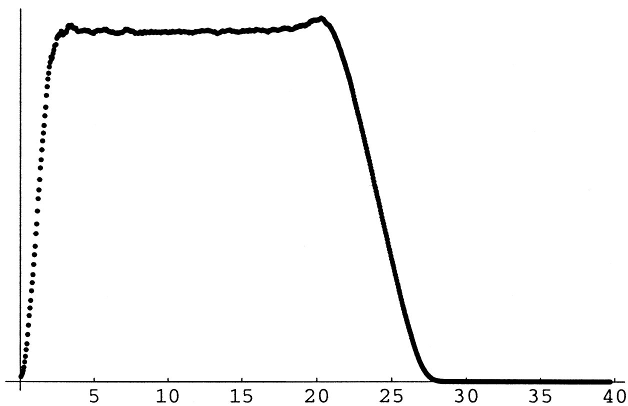

Fig. 4. Average two level correlation function. See text.

packed as closely as possible. In this case, if the levels have a density of $\frac{1}{2\pi}$, then the closest spacing between levels is $2\sqrt{\pi}/3^{1/4}$, which is approximately 2.7. For other arrangements of levels, the spacing will be slightly less. This length agrees quite well with the size of the shoulder. So, the shoulder length matches reasonably with the length scale expected from the particle spacing.

In principle, it is possible to compute the two-eigenvalue correlation function using SUSY techniques. However, it is very difficult, since the combination of non-Hermitian matrices and the requirement of self-duality leads to a complicated saddle-point manifold for the Q-matrices.

IV. CONCLUSION

In conclusion, we have considered an ensemble of strongly non-Hermitian, self-dual matrices. The two-level correlation function of this ensemble is particularly interesting, although the hoped for non-monotonicity has not emerged. It seems that all possible universality classes of non-Hermitian matrices are now known.

V. APPENDIX

We wish to derive Eq. (7) as an approximation to Eq. (6). Equation (6) may be derived following standard techniques for other ensembles.[15] The group of matrices X is a complex entension of the symplectic group, and the measure $[dX]$ is chosen invariant under multiplication by X in the group. As usual, M does not uniquely specify X. If X is right multiplied by any matrix Y such that $[Y, \Lambda]=0$, M is unchanged; generically there is an N parameter family of such matrices Y. These are matrices of the form $Y=e^P$ such that the only elements of P are immediately above and below the main diagonal:

$$P=\begin{pmatrix} 0 & p_1 & & & \\ p_1 & 0 & & & \\ & & 0 & p_2 & \\ & & p_2 & 0 & \\ & & & & \cdots \end{pmatrix} \tag{8}$$

Thus, we must choose a parametrization of X which avoids this problem. We choose to parametrize $X=e^H S$, where H are Hermitian *anti*-self-dual matrices ($ZH^TZ=H$) and where the elements of H immediately above and below the main diagonal vanish. S is a symplectic matrix. Equation (6) becomes, up to constant factors,

$$\prod_{i<j} |\lambda_i - \lambda_j|^8 \int [dH]\, J[H]\, e^{-1/2\, \mathrm{Tr}(\Lambda^\dagger e^{2H} \Lambda e^{-2H})} \tag{9}$$

where J is a Jacobian associated with this parametrization. Note that the integration over S drops out.

Expand e^H in powers of H to find

$$\prod_{i<j} |\lambda_i - \lambda_j|^8 \int [dH]\, J[H]\, e^{-1/2\, \mathrm{Tr}(4[H,\Lambda][H,\Lambda]^\dagger + O(H^4))} \tag{10}$$

The integrand is maximized for $H=0$. For $H=0$, the Jacobian is equal to unity and, ignoring the higher order terms in H, we obtain Eq. (7) after evaluating the Gaussian integral. Including the higher order terms in the exponential and including terms from the dependence of the Jacobian on H gives corrections to Eq. (7). Each correction at given order is a function of the positions of a certain number of eigenvalues λ_i, and is suppressed by factors of $1/|\lambda_i - \lambda_j|^2$.

REFERENCES

1. F. J. Dyson, *J. Math. Phys.* **3**:140 (1962).
2. J. J. M. Verbaarschot and I. Zahed, *Phys. Rev. Lett.* **70**:3853 (1993); J. J. M. Verbaarschot, *Phys. Rev. Lett.* **72**:2531 (1994).
3. A. Altland and M. R. Zirnbauer, *Phys. Rev. B* **55**:1142 (1997).
4. M. R. Zirnbauer, *J. Math. Phys.* **37**:4986 (1996).
5. P. W. Brouwer, C. Mudry, and A. Furusaki, *Nuc. Phys. B* **565**:653 (2000).
6. J. Ginibre, *J. Math. Phys.* **6**:440 (1965).
7. H.-J. Sommers, Y. V. Fyodorov, and M. Titov, preprint chao-dyn/9807015; Y. V. Fyodorov and H.-J. Sommers, *J. Math. Phys.* **38**:1918 (1997).
8. Y. V. Fyodorov, B. Khoruzhenko, and H.-J. Sommers, *Ann. Inst. H. Poincare, Phys. Theor.* **69**:449 (1998).
9. Y. V. Fyodorov, B. Khoruzhenko, and H.-J. Sommers, *Phys. Lett. A* **226**:46 (1997); Y. V. Fyodorov, B. Khoruzhenko, and H.-J. Sommers, *Phys. Lett. A* **79**:557 (1997).
10. J. Feinberg and A. Zee, preprint cond-mat/9703087.
11. K. B. Efetov, preprint cond-mat/9702091.
12. B. Jancovici, *Phys. Rev. Lett.* **46**:386 (1981).
13. J. M. Caillol *et al.*, *J. Stat. Phys.* **28**:325 (1982).
14. G. Tellez and P. J. Forrester, preprint cond-mat/9904388.
15. M. L. Mehta, *Random Matrices*, 1st edition (Academic Press, New York, 1967).
16. L. Girko, *Theor. Probab. Appl.* **29**:694 (1985); **30**:677 (1986).

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